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Green function for a charged spin-$\frac{1}{2}$ particle with anomalous magnetic moment in a plane-wave external electromagnetic field

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Abstract. The Green function for a charged spin-$\frac{1}{2}$ particle with anomalous magnetic moment in the presence of a plane-wave external electromagnetic field is calculated and shown to be simply related to the free-particle one.

1. Introduction

The Dirac equation for a charged spin-$\frac{1}{2}$ particle in an external plane-wave electromagnetic field was solved by Volkov [1] and the corresponding Green function was obtained by Schwinger [2]. Vaidya *et al* [3] and Vaidya and Hott [4] obtained by an algebraic method the relationship of this Green function with the free-particle one.

The solution of the Dirac–Pauli equation for a charged spin-$\frac{1}{2}$ particle with an anomalous magnetic moment in an external field of a somewhat general type (which includes the plane-wave field as a special case) was obtained by Chakrabarti [5]. Later Alan and Barut [6], Sen Gupta [7] and Melikian and Barber [8] considered the same problem. The Green function, however, has not been calculated before.

In this paper we calculate the Green function for a charged spin-$\frac{1}{2}$ particle with an anomalous magnetic moment in an external plane-wave electromagnetic field. We show that in this case the Green function is also related to that for a free particle in a simple manner. We also indicate how to solve the Dirac–Pauli equation exactly, thus obtaining a generalization of the Volkov solution.

This paper is organized as follows. In section 2 we formulate the problem and show how the charged particle problem can be reduced to the neutral particle one. In section 3 we obtain the Green function for the neutral particle by Schwinger's proper time method and show how it is related to the free-particle one. In section 4 the corresponding results for a charged particle are obtained. Section 5 contains a summary and a discussion of the results.

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2. Formulation of the problem

The Green function for a charged spin-$\frac{1}{2}$ particle with an anomalous magnetic moment in an external electromagnetic field F satisfies the equation (we use the notation of Bjorken and Drell [9]),

$$\left(\gamma^\mu\left(\mathrm{i}\frac{\partial}{\partial x'^\mu} - eA_\mu(x')\right) - a\sigma \cdot F(x') - m\right)G(x', x'') = \delta(x' - x'') \tag{1}$$

where $\sigma \cdot F(x') = \sigma^{\mu\nu}F_{\mu\nu}(x')$ and $F_{\mu\nu}(x') = \partial'_\nu A_\mu - \partial'_\mu A_\nu$ where $A^\mu(x')$ is the vector potential.

The parameter a is related to the anomalous magnetic moment of the particle. The total magnetic moment is $1 - 2a$ measured in units of $e\hbar/2mc$.

Writing

$$G(x', x'') = \langle x'|G|x''\rangle \tag{2}$$

where $x|x'\rangle = x'|x'\rangle$ we obtain

$$G = (\not{\pi} - a\sigma \cdot F - m)^{-1} \tag{3}$$

where $\pi_\mu = p_\mu - eA_\mu(x)$ and $[p_\mu, x_\nu] = \mathrm{i}g_{\mu\nu}$.

We restrict our attention to the case of a plane-wave field of the form [2]

$$F_{\mu\nu} = f_{\mu\nu}F(\xi) = f_{\mu\nu}\frac{\mathrm{d}A}{\mathrm{d}\xi} \tag{4}$$

where $\xi = n \cdot x$. The wavevector n and the numerical tensor $f^{\mu\nu}$ satisfy the equations

$$\begin{aligned} &n^2 = 0 \\ &n_\mu f^{\mu\nu} = 0 \\ &n_\mu {}^*f^{\mu\nu} = 0 \end{aligned} \tag{5}$$

where

$${}^*f^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\mu\alpha\beta}f_{\alpha\beta} \tag{6}$$

and $\epsilon^{0123} = 1$. In matrix notation (f^μ_ν is the μ–ν matrix element of f) we have with a choice of a normalization,

$$\begin{aligned} &(f^2)^\mu_\nu = n^\mu n_\nu \\ &({}^*f^2)^\mu_\nu = n^\mu n_\nu \\ &({}^*ff) = 0. \end{aligned} \tag{7}$$

Next using the relation

$$\gamma^\mu\sigma^{\alpha\beta} = \mathrm{i}(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha) - \epsilon^{\mu\nu\alpha\beta}\gamma_\nu\gamma_5 \tag{8}$$

we have

$$\not{n}\sigma \cdot f = 0. \tag{9}$$

Finally, the anticommutation relations

$$\{\sigma_{\mu\nu}, \sigma_{\alpha\beta}\}_+ = 2\mathrm{i}\epsilon_{\mu\nu\alpha\beta}\gamma_5 + 2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \tag{10}$$

give

$$(\sigma \cdot f)^2 = 0. \tag{11}$$

The problem formulated above can be simplified by using the results of Vaidya *et al* [3] and Vaidya and Hott [4] who have shown that for the external field of equation (4),

$$\not{\pi} = UV \not{p} (UV)^{-1}. \tag{12}$$

In the coordinate gauge the vector potential may be chosen as

$$A_\mu(x) = f_{\mu\nu}(x - x_0)^\nu \chi(\xi, \xi_0) \tag{13}$$

where

$$\chi(\xi, \xi_0) = \frac{A(\xi)}{\xi - \xi_0} - \frac{1}{(\xi - \xi_0)^2} \int_{\xi_0}^{\xi} A(\eta)\, \mathrm{d}\eta. \tag{14}$$

Here x_0 is an arbitrary reference point. Then one has

$$\begin{aligned} U(\xi, \xi_0) &= \exp \frac{-\mathrm{i}e}{n \cdot p} \left[\Gamma(\xi, \xi_0) A(\xi, \xi_0) \cdot p + \tfrac{1}{2} e \Omega(\xi, \xi_0) \right] \\ \Gamma \chi &= A(\xi) - (\xi - \xi_0)\chi \\ \Omega(\xi, \xi^0) &= \int_{\xi_0}^{\xi} A^2(\eta)\, \mathrm{d}\eta - \frac{1}{(\xi - \xi_0)} \left(\int_{\xi_0}^{\xi} A(\eta)\, \mathrm{d}\eta \right)^2. \end{aligned} \tag{15}$$

Also,

$$V(\xi, p) = \exp \left[\mathrm{i}e \frac{A(\xi) \sigma \cdot f}{4n \cdot p} \right]. \tag{16}$$

Equation (12) leads to the Green function for a spin-$\frac{1}{2}$ charged particle in an external plane-wave field in agreement with Schwinger's result.

Since

$$[UV, \sigma \cdot F] = 0 \tag{17}$$

it follows that

$$G = UV(\not{p} - a\sigma \cdot F - m)^{-1} (UV)^{-1}. \tag{18}$$

Thus the Green function for a charged spin-$\frac{1}{2}$ particle with an anomalous magnetic moment in a plane-wave field may be obtained from that of a neutral particle with an anomalous magnetic moment in the same field.

3. Calculation of the Green function for a neutral particle

In this section we calculate the Green function for a neutral spin-$\frac{1}{2}$ particle with an anomalous magnetic moment in an external plane-wave electromagnetic field F by Schwinger's proper time method. We also show that it is related to the free-particle Green function in a simple manner.

Writing G_0 in place of G in equation (3), we consider the integral representation

$$\begin{aligned} G_0 &= -\mathrm{i}(\not{p} - a\sigma \cdot F + m) \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}s(H+m^2)} \\ &= -\mathrm{i} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}s(H+m^2)} (\not{p} - a\sigma \cdot F + m) \end{aligned} \tag{19}$$

where

$$-H = (\not{p} - a\sigma \cdot F(x))^2. \tag{20}$$

For a plane wave, equations (8) and (11) give

$$-H = p^2 + 4aF(\xi)\, {}^*f_{\mu\nu} p^\mu \gamma^\nu \gamma_5. \tag{21}$$

Next, defining $\langle x', s| = \langle x'| \exp(-\mathrm{i}sH)$ the transformation function $\langle x', s|x'', 0\rangle$ satisfies the differential equation

$$\mathrm{i}\partial_s \langle x', s|x'', 0\rangle = \langle x', s|H|x'', 0\rangle. \tag{22}$$

To evaluate the matrix element on the right-hand side we must solve the equations of motion for the s(proper time) dependent operators and obtain the evolution operator H in an s-ordered form. This will be done next. In the following we omit the time argument for quantities at time zero.

The equations of motion are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} p^\mu(s) &= -4an^\mu F'(\xi(s))\, C(s) \\
\frac{\mathrm{d}}{\mathrm{d}s} \gamma_5(s) &= -8\mathrm{i}aF(\xi(s))\, C(s)\, \gamma_5(s) \\
\frac{\mathrm{d}}{\mathrm{d}s} \gamma^\mu(s) &= -8\mathrm{i}aF(\xi(s))\, {}^*f^{\mu\nu} p_\nu(s)\, \gamma_5(s) \\
\frac{\mathrm{d}}{\mathrm{d}s} x^\mu(s) &= 2p^\mu(s) + 4aF(\xi(s))\, {}^*f^{\mu\nu} \gamma_\nu(s)\, \gamma_5(s)
\end{aligned} \tag{23}$$

where we have defined

$$C(s) = {}^*f^{\mu\nu} p_\mu(s)\, \gamma_\nu(s)\, \gamma_5(s). \tag{24}$$

The operator C has the property

$$C(s)^2 = (n \cdot p(s))^2 \tag{25}$$

where as usual we do not write the unit matrix explicitly on the right-hand side. Using the equations of motion one can verify that C is a constant of motion. A simpler proof is the fact that it commutes with H. Obviously, we also have

$$n \cdot p(s) = n \cdot p \tag{26}$$

as confirmed by the first equation of motion. The last equation of the set gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} \xi(s) &= 2n \cdot p \\
\xi(s) &= \xi + 2n \cdot ps
\end{aligned} \tag{27}$$

which gives

$$[\xi(s), \xi] = 0. \tag{28}$$

Integrating the first equation of the set (23) gives

$$p^\mu(s) - p^\mu = -2a(n \cdot p)^{-1}(F(\xi(s) - F(\xi))\, Cn^\mu. \tag{29}$$

Next, defining

$$\eta(s) = -4\mathrm{i}aC(n \cdot p)^{-1}(A(\xi(s) - A(\xi)) \tag{30}$$

we obtain from the second and third equations

$$\begin{aligned} \gamma_5(s) &= \gamma_5 \exp[-\eta(s)] \\ \gamma^\mu(s) &= \gamma^\mu + {}^*f^{\mu\nu} p_\nu C^{-1}[\exp(\eta(s) - 1)]\,\gamma_5. \end{aligned} \tag{31}$$

Finally, the last equation of the set gives

$$\begin{aligned} x^\mu(s) - x^\mu &= 2p^\mu s + 4aC(n\cdot p)^{-1}F(\xi)\, sn^\mu \\ &\quad -\mathrm{i}/2(n\cdot p)^{-1}[n^\mu - {}^*f^{\mu\nu}\gamma_\nu\gamma_5 C(n\cdot p)^{-1}][1-\exp(-\eta)]. \end{aligned} \tag{32}$$

Using the last equation we obtain

$$2sC = {}^*f^{\mu\nu}(x(s)-x)_\mu\gamma_\nu\gamma_5 \tag{33}$$

and

$$4s^2p^2 = (x(s)-x)^2 - 16aCF(\xi)\,s^2. \tag{34}$$

Substitution of these results into equation (20) gives

$$-H = \frac{(x(s)-x)^2}{4s^2} \tag{35}$$

which may be put in an ordered form by using equation (32), which gives

$$[x^\mu(s), x_\mu] = 8\mathrm{i}s. \tag{36}$$

Hence

$$\mathrm{i}\partial_s\langle x', s|x'', 0\rangle = -\left[\frac{(x'-x'')^2}{4s^2} + \frac{2\mathrm{i}}{s}\right]\langle x', s|x'', 0\rangle. \tag{37}$$

Integrating the above equation we have

$$\langle x', s|x'', 0\rangle = \frac{-\mathrm{i}}{(4\pi s)^2}\exp\left[-\mathrm{i}\frac{(x'-x'')^2}{4s}\right]\Phi(x', x'') \tag{38}$$

where the multiplicative constant has been chosen to give the correct behaviour for small s ($\delta(x'-x'')$) except for the presence of Φ. The (matrix) function $\Phi(x', x'')$ may be determined by using equation (37) and the equations

$$\begin{aligned} \langle x', s|p^\mu(s)|x'', 0\rangle &= \mathrm{i}\partial'^\mu\langle x', s|x'', 0\rangle \\ \langle x', s|p^\mu|x'', 0\rangle &= -\mathrm{i}\partial''^\mu\langle x', s|x'', 0\rangle. \end{aligned} \tag{39}$$

Using equations (29), (32) and (38) we obtain

$$\mathrm{i}\partial'^\mu\Phi + \Phi 2a\left\langle\frac{C}{n\cdot p}\right\rangle F(\xi')\,n^\mu - \Phi\frac{\mathrm{i}}{2(\xi'-\xi'')}\zeta^\mu(1-\exp\langle\eta\rangle) = 0 \tag{40}$$

where we have defined

$$\begin{aligned} \left\langle\frac{C}{n\cdot p}\right\rangle &= \frac{{}^*f^{\mu\nu}(x'-x'')_\mu\gamma_\nu\gamma_5}{(\xi'-\xi'')} \\ \zeta^\mu &= n^\mu - {}^*f^{\mu\nu}\gamma_\nu\gamma_5\left\langle\frac{C}{n\cdot p}\right\rangle \\ \langle\eta\rangle &= -4\mathrm{i}a\left\langle\frac{C}{n\cdot p}\right\rangle(A(\xi') - A(\xi'')). \end{aligned} \tag{41}$$

After some effort one obtains

$$\partial'^{\mu}(\Phi\, \mathrm{e}^{\langle\eta\rangle/2}) = 0. \tag{42}$$

A similar calculation leads to

$$\partial''^{\mu}(\Phi\, \mathrm{e}^{\langle\eta\rangle/2}) = 0. \tag{43}$$

Hence

$$\Phi(x', x'') = \exp(2a\mathrm{i})\left[\frac{A(\xi') - A(\xi'')}{(\xi' - \xi'')}\,{}^{*}f^{\mu\nu}(x' - x'')_{\mu}\gamma_{\nu}\gamma_{5}\right]. \tag{44}$$

From the form of $\Phi(x', x'')$ it is clear that when s goes to zero the transformation function $\langle x', s|x'', 0\rangle$ goes to $\delta(x' - x'')$. Thus the Green function of a neutral spin-$\frac{1}{2}$ particle in a plane-wave external electromagnetic field is given by

$$\begin{aligned}G_0(x', x'') &= -(m + \mathrm{i}\gamma^{\mu}\partial'_{\mu} - a\sigma\cdot F(\xi'))\Phi(x', x'')\\ &\quad\times\int_0^{\infty}\mathrm{d}s\,\frac{\exp(-\mathrm{i}sm^2)}{(4\pi s)^2}\exp\left[-\mathrm{i}\frac{(x' - x'')^2}{4s}\right]\\ &= -\Phi(x', x'')\int_0^{\infty}\mathrm{d}s\,\frac{\exp(-\mathrm{i}sm^2)}{(4\pi s)^2}\\ &\quad\times\exp\left[-\mathrm{i}\frac{(x' - x'')^2}{4s}\right](m - \mathrm{i}\gamma^{\mu}\partial''_{\mu} - a\sigma\cdot F(\xi''))\end{aligned} \tag{45}$$

where in the last two lines the x''-derivatives act on all the x''-dependent terms on the left. The Green function can be rewritten in a more compact form as follows. We showed that

$$\langle x', s|x'', 0\rangle = \Phi(x', x'')\langle x'|\exp\left(\mathrm{i}sp^2\right)|x''\rangle. \tag{46}$$

Further, the form of the function $\Phi(x', x'')$ allows us to rewrite the last equation in the form

$$\langle x', s|x'', 0\rangle = W(x', \mathrm{i}\partial')\langle x'|\exp\left(\mathrm{i}sp^2\right)|x''\rangle W^{-1}(x'', \mathrm{i}\partial'') \tag{47}$$

where the derivatives act on $\langle x'|\exp\left(\mathrm{i}sp^2\right)|x''\rangle$ and we have defined

$$W(x, p) = \exp\left(\frac{2a\mathrm{i}}{n\cdot p}\right)CA(\xi). \tag{48}$$

Thus

$$\langle x', s|x'', 0\rangle = \langle x'|W(x, p)\exp\left(\mathrm{i}sp^2\right)W^{-1}(x, p)|x''\rangle. \tag{49}$$

In fact, it is trivial to see that

$$Wp^{\mu}W^{-1} = p^{\mu} + \frac{2an^{\mu}}{n\cdot p}CF(\xi) \tag{50}$$

so that

$$-H = (\not{p} - a\sigma\cdot F)^2 = W\not{p}^2W^{-1}. \tag{51}$$

Hence

$$\begin{aligned}G_0(x', x'') &= -\int_0^{\infty}\frac{\mathrm{d}s\,\exp(-\mathrm{i}sm^2)}{(4\pi s)^2}\\ &\quad\times W(x', \mathrm{i}\partial')\left(m + \frac{\gamma^{\mu}(x' - x'')_{\mu}}{2s}\right)\exp\left[-\mathrm{i}\frac{(x' - x'')^2}{4s}\right]W^{-1}(x'', -\mathrm{i}\partial'').\end{aligned} \tag{52}$$

Equation (51) suggests that one may directly relate the inverses of the operators $\not{p} - a\sigma \cdot F - m$ and $\not{p} - m$. One can verify that

$$W\gamma_\mu W^{-1} = \gamma_\mu + 2\frac{\mathrm{i}\sin[2aA(\xi)]}{n \cdot p}({}^*fp)_\mu\left[\cos[2aA(\xi)] + \frac{\mathrm{i}C}{n \cdot p}\sin[2aA(\xi)]\right]\gamma_5 \tag{53}$$

so that

$$W\not{n}W^{-1} = \not{n}. \tag{54}$$

Hence using equation (50)

$$W\not{p}W^{-1} = \not{p} + \frac{2a\not{n}C}{n \cdot p}F(\xi). \tag{55}$$

Since

$$\not{n}C = -\tfrac{1}{2}n \cdot p\,\sigma \cdot f \tag{56}$$

we obtain

$$W(\not{p} - m)^{-1}W^{-1} = (\not{p} - a\sigma \cdot F - m)^{-1}. \tag{57}$$

Although the above equation is true it would be difficult to obtain it without the use of the second-order formalism and the proper time method.

4. Calculation of the Green function for a charged particle

In section 2 we showed that the Green function for a charged fermion with an anomalous magnetic moment in a plane-wave electromagnetic field can be obtained from that of a neutral fermion with an anomalous magnetic moment in the same field. Now we have shown that the latter can be obtained from the free-particle Green function. Using equations (57) and (12) we obtain

$$G = UVW(\not{p} - m)^{-1}(UVW)^{-1}. \tag{58}$$

An equivalent form of equation above may be obtained by using the explicit form of the operator W. We have

$$W = \cos[2aA(\xi)] + \frac{\mathrm{i}C}{n \cdot p}\sin[2aA(\xi)]. \tag{59}$$

We may define the operators D_+ and D_- where

$$\begin{aligned}(\not{p} - m)\sigma \cdot f &= 2(D_+ - C)\\ \sigma \cdot f(\not{p} - m) &= 2(D_- - C).\end{aligned} \tag{60}$$

Thus

$$\begin{aligned}D_+ &= -\mathrm{i}f^{\mu\nu}p_\nu\gamma_\mu - \tfrac{1}{2}m\sigma \cdot f\\ D_- &= \mathrm{i}f^{\mu\nu}p_\nu\gamma_\mu - \tfrac{1}{2}m\sigma \cdot f.\end{aligned} \tag{61}$$

It is easy to verify that

$$T_-(\not{p} - m)^{-1}T_+{}^{-1} = (\not{p} - a\sigma \cdot F - m)^{-1} \tag{62}$$

where

$$\begin{aligned}T_+ &= \exp\left[\frac{2a\mathrm{i}D_+A(\xi)}{n \cdot p}\right]\\ T_- &= \exp\left[\frac{2a\mathrm{i}D_-A(\xi)}{n \cdot p}\right].\end{aligned} \tag{63}$$

Thus

$$G = UVT_-(\not{p} - m)^{-1}UVT_+{}^{-1}. \tag{64}$$

5. Summary and discussion

In this paper we have obtained the Green function for a charged spin-$\frac{1}{2}$ particle with an anomalous magnetic moment in an external plane-wave electromagnetic field. We also showed how it is related to the free-particle Green function. We considered the case of plane polarization. Corresponding results for the case of arbitrary polarization are easily obtained.

We can use our results to construct a complete set of solutions of the Dirac–Pauli equation in the form

$$\Psi(x) = UVW(x, \mathrm{i}\partial)\Psi_0(x). \tag{65}$$

where $\Psi_0(x)$ satisfies the free Dirac equation. Thus we have obtained a generalization of the Volkov solution. An alternative form of the solution is obtained if we use equation (62), and change the signs of a and m to obtain

$$T_+(\not{p} - m)T_-{}^{-1} = \not{p} - m - a\sigma \cdot F \tag{66}$$

so that

$$\Psi(x) = UVT_-(x, \mathrm{i}\partial)\Psi_0(x) \tag{67}$$

where the equivalence of the two forms of the solution becomes evident when we use equation (60). Equation (67) corresponds to the result of Alan and Barut [6] when the Weyl representation for the Dirac matrices is used.

As a final remark we observe that equation (65) also suggests that our use of the formal operator $(n \cdot p)^{-1}$ is justified by the observation that it needs to be well defined on the solutions of the free Dirac equation. This is true for the $m \neq 0$ case.

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